

# Analogies between the geodetic number and the Steiner number of some classes of graphs

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## Abstract

A set of vertices  $S$  of a graph  $G$  is a geodetic set of  $G$  if every vertex  $v \notin S$  lies on a shortest path between two vertices of  $S$ . The minimum cardinality of a geodetic set of  $G$  is the geodetic number of  $G$  and it is denoted by  $g(G)$ . A Steiner set of  $G$  is a set of vertices  $W$  of  $G$  such that every vertex of  $G$  belongs to the set of vertices of a connected subgraph of minimum size containing the vertices of  $W$ . The minimum cardinality of a Steiner set of  $G$  is the Steiner number of  $G$  and it is denoted by  $s(G)$ . Let  $G$  and  $H$  be two graphs and let  $n$  be the order of  $G$ . The corona product  $G \odot H$  is defined as the graph obtained from  $G$  and  $H$  by taking one copy of  $G$  and  $n$  copies of  $H$  and joining by an edge each vertex from the  $i^{th}$ -copy of  $H$  with the  $i^{th}$ -vertex of  $G$ . We study the geodetic number and the Steiner number of corona product graphs. We show that if  $G$  is a connected graph of order  $n \geq 2$  and  $H$  is a non complete graph, then  $g(G \odot H) \leq s(G \odot H)$ , which partially solve the open problem presented in [*Discrete Mathematics* **280** (2004) 259–263] related to characterize families of graphs  $G$  satisfying that  $g(G) \leq s(G)$ .

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# 1 Introduction

The Steiner distance of a set of vertices of a graph was introduced as a generalization of the distance between two vertices [3]. In this sense, Steiner sets in graphs could be understood as a generalization of geodetic sets in graphs. Nevertheless, its relationship is not exactly obvious. Some of the primary results in this topic were presented in [4], where the authors tried to obtain a result relating geodetic sets and Steiner sets. That is, they tried to show that every Steiner set of a graph is also a geodetic set. Fortunately, the author of [9] showed by a counterexample that not every Steiner set of a graph is a geodetic set, and it was pointed out an open question related to characterizing those graphs satisfying that every Steiner set is geodetic or vice versa. Some relationships between Steiner sets and geodetic sets were obtained in [2, 4, 7, 8, 9]. For instance, [2] was dedicated to obtain some families of graphs in which every Steiner set is a geodetic set, but the problem of characterizing such a graphs remains open.

In this work we show some classes of graph in which every Steiner set is a geodetic set. For instance, we prove that if  $G$  is a graph with diameter two, then every Steiner set of  $G$  is also a geodetic set. We also obtain some relationships between the Steiner (geodetic) sets of corona product graphs and the Steiner (geodetic) sets of its factors and, as a consequence of this study, we obtain that if  $G$  is a corona product graph, then every Steiner set of  $G$  is a geodetic set.

We begin by stating some terminology and notation. In this paper  $G = (V, E)$  denotes a connected simple graph of order  $n = |V|$ . We denote two adjacent vertices  $u$  and  $v$  by  $u \sim v$ . Given a set  $W \subset V$  and a vertex  $v \in V$ ,  $N_W(v)$  represents the set of neighbors that  $v$  has in  $W$ , i.e.  $N_W(v) = \{u \in W : u \sim v\}$ . The subgraph induced by a set  $W \subset V$  will be denoted by  $\langle W \rangle$ .

The distance  $d_G(u, v)$  between two vertices  $u$  and  $v$  is the length of a shortest  $u - v$  path in  $G$ . If there is no ambiguity, we will use the notation  $d(u, v)$  instead of  $d_G(u, v)$ . A shortest  $u - v$  path is called  $u - v$  *geodesic*. We define  $I_G[u, v]$ <sup>1</sup> to be the set of all vertices lying on some  $u - v$  geodesic of  $G$ , and for a nonempty set  $S \subseteq V$ ,  $I_G[S] = \bigcup_{u, v \in S} I_G[u, v]$  ( $I[S]$  for short). A set  $S \subseteq V$  is a *geodetic set* of  $G$  if  $I_G[S] = V$  and a geodetic set of minimum cardinality is called a *minimum geodetic set* [6]. The cardinality

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<sup>1</sup>If there is no ambiguity, then we will use  $I[u, v]$ .

of a minimum geodetic set of  $G$  is called the *geodetic number* of  $G$  and it is denoted by  $g(G)$ . A vertex  $v \in V$  is *geodominated* by a pair  $x, y \in V$  if  $v$  lies on an  $x - y$  geodesic of  $G$ . For an integer  $k \geq 2$ , a vertex  $v$  of a graph  $G$  is *k-geodominated* by a pair  $x, y$  of vertices in  $G$  if  $d(x, y) = k$  and  $v$  lies on an  $x - y$  geodesic of  $G$ . A subset  $S \subseteq V$  is a *k-geodetic set* if each vertex  $v$  in  $\bar{S} = V - S$  is *k-geodominated* by some pair of vertices of  $S$ . The minimum cardinality of a *k-geodetic set* of  $G$  is its *k-geodetic number*  $g_k(G)$ . It is clear that  $g(G) \leq g_k(G)$  for every  $k$ .

For a nonempty set  $W$  of vertices of a connected graph, the *Steiner distance*  $d(W)$  of  $W$  is the minimum size of a connected subgraph of  $G$  containing  $W$  [3]. Necessarily, such a subgraph is a tree and it is called a *Steiner tree* with respect to  $W$  or a *Steiner  $W$ -tree*, for short. For a set  $W \subseteq V$ , the set of all vertices of  $G$  lying on some Steiner  $W$ -tree is denoted by  $S_G[W]$  (or by  $S[W]$ , if there is no ambiguity). If  $S_G[W] = V$ , then  $W$  is called a *Steiner set* of  $G$ . The *Steiner number* of a graph  $G$ , denoted by  $s(G)$ , is the minimum cardinality among the Steiner sets of  $G$ .

Let  $G$  and  $H$  be two graphs and let  $n$  be the order of  $G$ . The corona product  $G \odot H$  is defined as the graph obtained from  $G$  and  $H$  by taking one copy of  $G$  and  $n$  copies of  $H$  and then joining by an edge, all the vertices from the  $i^{th}$ -copy of  $H$  with the  $i^{th}$ -vertex of  $G$ . Throughout the article we will denote by  $V = \{v_1, v_2, \dots, v_n\}$  the set of vertices of  $G$  and by  $H_i = (V_i, E_i)$  the copy of  $H$  in  $G \odot H$  such that  $v_i \sim v$  for every  $v \in V_i$ .

## 2 Geodetic number of corona product graphs

We begin by stating some results that we will use as tool in this section. The first one is the following well-known result.

**Lemma 1.** [6] *Let  $G$  be a connected graph of order  $n$ . Then  $g(G) = n$  if and only if  $G \cong K_n$ .*

Our second tool will be the following useful lemma related to the geodetic sets of corona product graphs.

**Lemma 2.** *Let  $G = (V, E)$  be a connected graph of order  $n$  and let  $H$  be a graph. Let  $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \dots, H_n = (V_n, E_n)$  be the  $n$  copies of  $H$  in  $G \odot H$ .*

- (i) *Given three different vertices  $a, b$  and  $v$  of  $G \odot H$ , if  $v \in V_i$  and ( $a \notin V_i$  or  $b \notin V_i$ ), then  $v \notin I_{G \odot H}[a, b]$ .*

- (ii) If  $W$  is a geodetic set of  $G \odot H$ , then  $W \cap V_i \neq \emptyset$ , for every  $i \in \{1, \dots, n\}$ .
- (iii) If  $W$  is a minimum geodetic set of  $G \odot H$  and either  $n \geq 2$  or ( $n = 1$  and  $H$  is a non-complete graph), then  $W \cap V = \emptyset$ .
- (iv) If  $H$  is a non-complete graph and  $W$  is a minimum geodetic set of  $G \odot H$ , then for every  $i \in \{1, \dots, n\}$ ,  $W_i = W \cap V_i$  is a geodetic set of  $\langle v_i \rangle \odot H_i$ .

*Proof.* (i) and (ii) follow directly from the fact that the vertices belonging to  $V_i$  are adjacent to only one vertex not in  $V_i$ .

Now let  $W'$  be a geodetic set of  $G \odot H$  and let  $W = W' - V$ . We will show that  $W$  is a geodetic set of  $G \odot H$ . By (ii) we have that for every  $i \in \{1, \dots, n\}$  it is satisfied,  $W \cap V_i \neq \emptyset$ , and by (i), we have that if  $v \in V_i$ , then there exist  $a', b' \in V_i \cap W$ , such that  $v \in I_{G \odot H}[a', b']$ . Now, if  $n \geq 2$ , then for every vertex  $v_i \in V$  we have that  $v_i \in I_{G \odot H}[c, d]$ , with  $c \in W \cap V_i$  and  $d \in W \cap V_j$ ,  $j \neq i$ . Thus,  $W$  is a geodetic set of  $G \odot H$ . On the other hand, if  $n = 1$  and  $H$  is a non-complete graph, then  $G \odot H$  is a non-complete graph and, by Lemma 1,  $g(G \odot H) \leq n_2$ , where  $n_2$  is the order of  $H$ . Hence,  $\langle W \rangle$  is not isomorphic to a complete graph. So, there exist two vertices  $x, y \in W$  such that the vertex of  $G \cong K_1$  belongs to  $I_{G \odot H}[x, y]$ . Moreover, by (i), every vertex of  $\overline{W}$  different from the vertex of  $G$  is geodominated by two vertices of  $W$ . Thus,  $W$  is a geodetic set of  $G \odot H$  and, as a consequence, (iii) follows.

Finally, let  $H$  be a non-complete graph and let  $W$  be a minimum geodetic set of  $G \odot H$ . By (ii) we have that  $W_i = W \cap V_i \neq \emptyset$ . Also, by (iii) we have that  $V \cap W = \emptyset$ . Now we suppose that  $W_i$  is not a geodetic set of  $\langle v_i \rangle \odot H_i$ . Hence, there exists  $v \in V_i \cup \{v_i\}$  such that  $v \notin I_{\langle v_i \rangle \odot H_i}[x, y]$  for every  $x, y \in W_i$ . By (i) we have that if  $v \in V_i - W$ , then  $v$  must be geodominated by vertices of  $W_i$ , which is a contradiction, so  $v \notin V_i$ , i.e.,  $v = v_i$ . Now, since  $v_i$  is adjacent to every vertex of  $H_i$  and  $H_i$  is a non-complete graph, we obtain that there exist two non-adjacent vertices  $c, d$  of  $H_i$  such that  $c, d \in W_i$ . Hence,  $v_i \in I_{\langle v_i \rangle \odot H_i}[c, d]$ , a contradiction. Therefore, (iv) follows.  $\square$

The following relation between  $g(H)$  and  $g(K_1 \odot H)$ , which we will use here, was obtained in [2].

**Lemma 3.** [2] For any graph  $H$ ,  $g(K_1 \odot H) \geq g(H)$ .

A vertex  $v$  is an extreme vertex in a graph  $G$  if the subgraph induced by its neighbors is complete. The following lemma is a consequence of the

observation that each extreme vertex  $v$  of  $G$  is either the initial or terminal vertex of a geodesic containing  $v$ .

**Lemma 4.** [1] *Every geodetic set of a graph contains its extreme vertices.*

**Proposition 5.** *Let  $G$  be a connected graph of order  $n_1$  and let  $H$  be a graph of order  $n_2$ . If  $n_1 \geq 2$  or ( $n_1 = 1$  and  $H$  is a non-complete graph), then*

$$n_1 g(H) \leq g(G \odot H) \leq n_1 n_2.$$

*The upper bound is achieved if and only if  $H$  is isomorphic to a graph in which every connected component is isomorphic to a complete graph.*

*Moreover, if no connected component of  $H$  is isomorphic to a complete graph, then*

$$g(G \odot H) \leq n_1(n_2 - 1).$$

*Proof.* If  $H \cong K_{n_2}$ , the vertices of the set  $\cup_{i=1}^{n_1} V_i$  are extreme vertices. Then, by Lemma 4 we have  $g(G \odot K_{n_2}) \geq n_1 n_2 = n_1 g(K_{n_2})$ . For non-complete graphs the lower bound follows directly from Lemma 2 (iv) and Lemma 3. On the other hand, if  $n_1 \geq 2$ , then every vertex  $v_i \in V$  is geodominated, in  $G \odot H$ , by two vertices belonging to different copies of  $H$ . So, the set  $\cup_{i=1}^{n_1} V_i$  is a geodetic set of  $G \odot H$ . Thus,  $g(G \odot H) \leq n_1 n_2$ . Finally, if  $n_1 = 1$ , then the order of  $G \odot H$  is  $n_2 + 1$ . Hence, if  $H$  is a non-complete graph, then Lemma 1 leads to the upper bound  $g(K_1 \odot H) \leq n_2$ .

Now, let us suppose that there is a component of  $H$  which is not isomorphic to a complete graph. In such a case, there are three different vertices  $u_i, x_i, y_i \in V_i$  such that  $u_i \in I_{H_i}[x_i, y_i]$ , with  $i \in \{1, \dots, n_1\}$ . Let  $V = \{v_1, \dots, v_{n_1}\}$ ,  $U_i = V_i - \{u_i\}$ , with  $i \in \{1, \dots, n_1\}$ , and let  $U = \cup_{i=1}^{n_1} U_i$ . We will show that  $U$  is a geodetic set of  $G \odot H$ . Since for every vertex  $u_i \in \overline{U}_i$  we have that  $u_i \in I_{H_i}[x_i, y_i]$ , we obtain that  $u_i \in I_{G \odot H}[U]$ . Also, as for every  $v_i \in V$ , we have that  $v_i \in I_{G \odot H}[a, b]$ , for some  $a \in U_i$  and  $b \in U_j$ , with  $i \neq j$ , we obtain that  $v_i \in I_{G \odot H}[U]$ . Therefore,  $U$  is a geodetic set of  $G \odot H$  and, as a consequence,  $g(G \odot H) \leq |U| = n_1(n_2 - 1)$ . Therefore, if  $g(G \odot H) = n_1 n_2$ , then  $H$  is isomorphic to a graph in which every connected component is isomorphic to a complete graph.  $\square$

**Theorem 6.** *Let  $G$  be a connected graph of order  $n$  and let  $H$  be a non-complete graph. Then,*

$$g(G \odot H) = n g(K_1 \odot H).$$

*Proof.* Let  $W$  be a minimum geodetic set of  $G \odot H$ . From Lemma 2 (iii) we have that  $W \cap V = \emptyset$ . Also, by Lemma 2 (ii) and (iv) we have that for every  $i \in \{1, \dots, n\}$ , the set  $W_i = W \cap V_i \neq \emptyset$  is a geodetic set of  $\langle v_i \rangle \odot H_i \cong K_1 \odot H$ . Hence, we have

$$g(G \odot H) = |W| = \sum_{i=1}^n |W_i| \geq \sum_{i=1}^n g(\langle v_i \rangle \odot H_i) = ng(K_1 \odot H).$$

On the other hand, let  $U_i \subset V_i \cup \{v_i\}$  be a minimum geodetic set of  $\langle v_i \rangle \odot H_i$  and let  $U = \cup_{i=1}^n U_i$ . Notice that, by Lemma 2 (iii),  $v_i \notin U_i$ . We will show that  $U$  is a geodetic set of  $G \odot H$ . Let us consider a vertex  $x$  of  $G \odot H$ . We have the following cases.

Case 1: If  $x \in (V_i \cup \{v_i\}) - U_i$ , then there exist  $u, v \in U_i$  such that  $x \in I_{\langle v_i \rangle \odot H_i}[u, v]$ . So,  $x \in I_{G \odot H}[u, v]$ .

Case 2: If  $x = v_i \in V$  and  $n \geq 2$ , then for every vertex  $v \in U_i$  and some  $u \in U_j$ ,  $j \neq i$  we have that  $x \in I_{G \odot H}[u, v]$ . Also, if  $x \in V$  and  $n = 1$ , then as  $H$  is a non-complete graph, there exist two different vertices  $a, b \in U = U_1$ , such that  $x \in I_{G \odot H}[a, b]$ .

Thus, every vertex  $x$  of  $G \odot H$  is geodominated by a pair of vertices of  $U$  and, as a consequence,  $g(G \odot H) \leq ng(K_1 \odot H)$ . Therefore, we obtain that  $g(G \odot H) = ng(K_1 \odot H)$ .  $\square$

The geodetic number of wheel graphs and fan graphs were studied in [2] and [5].

**Remark 7.** [2] If  $n \geq 4$ , then  $g(W_{1,n}) = \lceil \frac{n}{2} \rceil$ .

**Remark 8.** [2, 5] If  $n \geq 3$ , then  $g(F_{1,n}) = \lceil \frac{n+1}{2} \rceil$ .

As a particular cases of Theorem 6 and by using the above remarks we obtain the following results.

**Corollary 9.** Let  $G$  be a connected graph of order  $n_1$ .

- (i) If  $n_2 \geq 4$ , then  $g(G \odot C_{n_2}) = n_1 g(W_{1,n_2}) = n_1 \lceil \frac{n_2}{2} \rceil$ .
- (ii) If  $n_2 \geq 3$ , then  $g(G \odot P_{n_2}) = n_1 g(F_{1,n_2}) = n_1 \lceil \frac{n_2+1}{2} \rceil$ .

From Lemma 3 we have that  $g(K_1 \odot H) \geq g(H)$ . Hence, Theorem 6 leads to the lower bound of Proposition 5. Now we are interested in those graphs in which  $g(H) = g(K_1 \odot H)$ .

**Theorem 10.** *For a connected graph  $H$ , the following statements are equivalent:*

- $g(H) = g(K_1 \odot H)$ .
- $g(H) = g_2(H)$ .

*Proof.* Let us suppose  $g(H) = g_2(H)$ . Let  $W$  be a 2-geodetic set of minimum cardinality in  $H$ . Hence, for every vertex  $u \in \overline{W}$  there exist  $a, b \in W$ , such that  $u \in I_H[a, b]$  and  $d_H(a, b) = 2$ . Since every geodesic of length two in  $H$  is a geodesic in  $K_1 \odot H$ , we have that  $W$  is a geodetic set of  $K_1 \odot H$ . As a consequence,  $g(H) \geq g(K_1 \odot H)$ . Hence, by Lemma 3 we conclude that  $g(H) = g(K_1 \odot H)$ .

On the other hand, let us suppose  $g(H) = g(K_1 \odot H)$ . Let  $U$  be a minimum geodetic set of  $K_1 \odot H$  and let  $v$  be the vertex of  $K_1$ . Since  $H$  can not be a complete graph, by Lemma 2 (iii) we have that  $v \notin U$ . Now, since  $K_1 \odot H$  has diameter two, we have that for every vertex  $u$  of  $H$  not belonging to  $U$ , there exist  $a, b \in U$  such that  $u \in I_{K_1 \odot H}[a, b]$  and  $d_H(a, b) = 2$  (Note that if  $d_H(a, b) > 2$ , then  $u \notin I_{K_1 \odot H}[a, b] = \{a, b, v\}$ ). Hence,  $U$  is a 2-geodetic set of  $H$ . Thus,  $g_2(H) \leq |U| = g(K_1 \odot H) = g(H)$ . Also, as  $g(H) \leq g_2(H)$ , we obtain that  $g(H) = g_2(H)$ .  $\square$

**Theorem 11.** *Let  $G$  be a connected graph of order  $n$  and let  $H$  be a connected non-complete graph. Then the following statements are equivalent:*

- $g(G \odot H) = ng(H)$ .
- $g(H) = g_2(H)$ .

*Proof.* The result is a direct consequence of Theorem 6 and Theorem 10.  $\square$

Since for every graph  $H$  of diameter two we have  $g(H) = g_2(H)$ , Theorem 11 leads to the following result.

**Corollary 12.** *Let  $G$  be a connected graph of order  $n$  and let  $H$  be a graph. If  $D(H) = 2$ , then*

$$g(G \odot H) = ng(H).$$

Another consequence of Theorem 10 is the following result.

**Corollary 13.** *Let  $G$  and  $H$  be two connected graphs of order  $n_1$  and  $n_2$ , respectively. Let  $N_k$  be the empty graph of order  $k \geq 2$ . Then*

$$g(G \odot (H \odot N_k)) = n_1 n_2 k.$$

*Proof.* The result follows from the fact that  $g(H \odot N_k) = g_2(H \odot N_k) = n_2 k$ . That is, the set composed by the  $n_2 k$  pendant vertices of  $H \odot N_k$  form a geodetic set of  $H \odot N_k$  which is a 2-geodetic set. So,  $g(H \odot N_k) \leq g_2(H \odot N_k) \leq n_2 k$ . Moreover, since every pendant vertex is an extreme vertex, by Lemma 4 we have  $g(H \odot N_k) \geq n_2 k$ . Therefore, the result follows.  $\square$

The following result improves the lower bound in Proposition 5 for those graphs whose geodetic number is different from its 2-geodetic number.

**Theorem 14.** *Let  $G$  be a connected graph of order  $n$  and let  $H$  be a non-complete graph. If  $g(H) \neq g_2(H)$ , then*

$$g(G \odot H) \geq n(g(H) - 1).$$

*Proof.* As a direct consequence of Theorem 10 and Lemma 3 we obtain that, if  $g(H) \neq g_2(H)$ , then

$$g(K_1 \odot H) \geq g(H) - 1. \tag{1}$$

Hence, the result follows directly by Theorem 6 and (1).  $\square$

### 3 Steiner number of corona product graphs

In this section the main tool will be the following basic lemma.

**Lemma 15.** *Let  $G = (V, E)$  be a connected graph of order  $n_1$  and let  $H$  be a graph of order  $n_2$ . Let  $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \dots, H_n = (V_n, E_n)$  be the  $n_1$  copies of  $H$  in  $G \odot H$ .*

- (i) *If  $n_1 \geq 2$  and  $A \subseteq \cup_{i=1}^{n_1} V_i$  with  $A \cap V_i \neq \emptyset$ , for every  $i \in \{1, \dots, n_1\}$ , then every Steiner  $A$ -tree contains all vertices of  $G$*
- (ii) *If  $U$  is a Steiner set of  $G \odot H$ , then  $U \cap V_i \neq \emptyset$ , for every  $i \in \{1, \dots, n\}$ .*
- (iii) *If  $n_1 \geq 2$  or  $n_2 \geq 2$ , then for every Steiner set  $U$  of minimum cardinality in  $G \odot H$  it follows  $U \cap V = \emptyset$ .*



*Proof.* (i) follows from the fact that if there exists a Steiner  $A$ -tree  $T$  not containing a vertex of  $G$ , then  $T$  is not connected, which is a contradiction. (ii) follows directly from the fact that the vertices belonging to  $V_i$  are adjacent to only one vertex not in  $V_i$ .

Now let  $U'$  be a Steiner set of  $G \odot H$  and let  $U = U' - V$ . We will show that  $U$  is a Steiner set for  $G \odot H$ . By (ii) we have that  $U \cap V_i \neq \emptyset$ , for every  $i \in \{1, \dots, n\}$ . Also, if  $v \in V_i$ , then we have that there exists a Steiner  $U$ -tree in  $G \odot H$  such that it contains the vertex  $v$ . Now, since  $n_1 \geq 2$  we obtain that every vertex  $v_i \in V$  belongs to every Steiner  $U$ -tree (note that every shortest  $u - v$  path, where  $v \in V_i$  and  $u \in V_j$ ,  $j \neq i$ , must contain  $v_i$ ). Thus,  $U$  is a Steiner set for  $G \odot H$  and (iii) follows.  $\square$

The next lemmas obtained in [4] will be useful to obtain our results.

**Lemma 16.** [4] *Let  $G$  be a connected graph of order  $n$ . Then  $s(G) = n$  if and only if  $G \cong K_n$ .*

Before present our main results about the Steiner number, let us show the following useful lemma.

**Lemma 17.** *For any graph  $G$ ,  $s(K_1 \odot G) \geq s(G)$ .*

*Proof.* Let  $n$  be the order of  $G$ . If  $G \cong K_n$ , then  $K_1 \odot G \cong K_{n+1}$ , so by Lemma 16,  $s(K_1 \odot G) = n+1 > n = s(G)$ . If  $G \not\cong K_n$ , then the result follows immediately from Lemma 15 (iii).  $\square$

**Proposition 18.** *Let  $G = (V, E)$  be a connected graph of order  $n_1$  and let  $H$  be a graph of order  $n_2$ . If  $n_1 \geq 2$ , then  $s(G \odot H) = n_1 n_2$ .*

*Proof.* Let  $A = \cup_{i=1}^{n_1} V_i$ . By Lemma 15 (iii) we have that every Steiner set of minimum cardinality is a subset of  $A$ . Thus,  $A$  is a Steiner set of  $G \odot H$  and, as a consequence,  $s(G \odot H) \leq n_1 n_2$ .

Now, let us suppose  $B$  is a Steiner set of minimum cardinality in  $G \odot H$ . By Lemma 15 (iii) we have that  $B$  does not contain any vertex of  $G$ . Now, let us suppose there exists a vertex  $v_i \in V$  such that  $B \cap V_i \subsetneq V_i$ . Let  $B_i = B \cap V_i$  and let  $u \in V_i - B_i$ . Since every vertex of  $B_i$  is adjacent to  $v_i$ , and  $v_i$  belongs to every Steiner  $B$ -tree  $T$ , we have that the size of the restriction of  $T$  to  $V_i \cup \{v_i\}$  is  $|B_i|$ . Thus, the vertex  $u$  does not belong to any Steiner  $B$ -tree in  $G \odot H$ , which is a contradiction. Thus, for every  $i \in \{1, \dots, n_1\}$  we have that  $B \cap V_i = V_i$ . Therefore,  $s(G \odot H) \geq n_1 n_2$ . The proof is complete.  $\square$

The Steiner number of wheel graphs and fan graphs were studied in [2] and [5].

**Remark 19.** [2] *If  $n \geq 4$ , then  $s(W_{1,n}) = n - 2$ .*

**Remark 20.** [2, 5] *If  $n \geq 3$ , then  $g(F_{1,n}) = n - 1$ .*

**Theorem 21.** *Let  $H$  be a connected non complete graph. Then the following statements are equivalent:*

- $s(K_1 \odot H) = s(H)$ .
- $D(H) = 2$ .

*Proof.* Let  $B$  be a Steiner set of minimum cardinality in  $H$  and let  $v$  be the vertex of  $K_1$ . If  $D(H) = 2$ , then there exist three vertices of  $H$  such that  $x, y \in B$  and  $z \notin B$ ,  $d_H(x, y) = 2$  and  $x, y \in N_B(z)$ . So, if we take a Steiner  $B$ -tree  $T$  in  $H$  containing the path  $xzy$ , then replacing the vertex  $z$  of  $T$  by the vertex  $v$ , and replacing every edge  $uz$  of  $T$  by a new edge  $uv$ , we obtain a Steiner  $B$ -tree  $T'$  in  $K_1 \odot H$ . Hence,  $B$  is a Steiner set for  $K_1 \odot H$ . Therefore,  $s(H) \geq s(K_1 \odot H)$  and, by Lemma 17, we conclude  $s(H) = s(K_1 \odot H)$ .

Now, let  $H$  be a graph such that  $s(K_1 \odot H) = s(H)$ . Let  $W$  be a Steiner set of minimum cardinality in  $K_1 \odot H$  and let  $v$  be the vertex of  $K_1$ . We first show that  $W$  is a Steiner set for  $H$ . Note that by Lemma 15 (iii),  $v \notin W$ . Since the star graph of center  $v$  is a Steiner  $W$ -tree, we have that the Steiner distance of  $W$  in  $K_1 \odot H$  is  $d(W) = |W|$ . If  $\langle W \rangle$  is connected, then  $|W|$  is the order of  $K_1 \odot H$ , which is a contradiction. Thus,  $\langle W \rangle$  is non connected. Let  $\langle W_1 \rangle, \langle W_2 \rangle, \dots, \langle W_k \rangle$  be the connected components of  $\langle W \rangle$ . If there exists a vertex  $u$  of  $H$  such that  $u \notin W$  and  $N_{W_i}(u) = \emptyset$ , for some  $i$ , then the Steiner distance of  $W$  in  $K_1 \odot H$  is  $d(W) > \sum_{i=1}^k |W_i| = |W|$ , which is a contradiction. So, every vertex  $u$  of  $H$  not belonging to  $W$  is at distance one to every connected component of  $\langle W \rangle$  and, as a consequence,  $W$  is a Steiner set of  $H$ , which has minimum cardinality since  $s(K_1 \odot H) = s(H)$ . Let us show that  $D(H) = 2$ . On the contrary, we suppose that  $D(H) \geq 3$  (note that  $H$  is not a complete graph). From the assumption  $D(H) \geq 3$ , we conclude that for each vertex  $u$  of  $H$ , not belonging to  $W$ , there exist  $y \in W_i$  (for some  $i$ ) such that  $d(y, u) = 2$ . Let  $x \in W_i$  be a neighbor of both  $u$  and  $y$ , and let  $W' = W - \{x\}$ . Then we have that every Steiner  $W$ -tree of  $H$  is a Steiner  $W'$ -tree of  $H$  and, as a consequence,  $W'$  is a Steiner set of  $H$ , which is a contradiction. Therefore,  $D(H) = 2$ .  $\square$

## 4 Relationships between the geodetic number and the Steiner number

Here we show some classes of graphs where the Steiner number is greater than or equal to the geodetic number.

**Theorem 22.** *If  $G$  is a graph of diameter two, then every Steiner set for  $G$  is a geodetic set for  $G$ .*

*Proof.* Let  $W$  be a Steiner set of minimum cardinality in  $G$  and let  $n$  be the order of  $G$ . If  $\langle W \rangle$  is connected, then  $|W| = n$ . So, by Lemma 16 we have that  $G \cong K_n$ , which is a contradiction because  $G$  has diameter two. Thus,  $\langle W \rangle$  is non connected. Let  $\langle B_1 \rangle, \langle B_2 \rangle, \dots, \langle B_r \rangle$  be the connected components of  $W$ . We assume that  $W$  is not a geodetic set. Then there exists a vertex  $x$  of  $G$  such that  $x \notin I[W]$ . Thus,  $x \notin W$  and  $x \notin I[u, v]$  for every  $u, v \in W$ . Hence,  $N_W(x) \subseteq B_i$ , for some  $i \in \{1, \dots, r\}$ . Since  $G$  has diameter two, any Steiner  $W$ -tree is formed by  $r$  Steiner  $B_i$ -trees connected by vertices  $v_1, v_2, \dots, v_t$ ,  $t \geq 1$ , not belonging to  $W$  such that  $N_W(v_i) \not\subseteq B_j$ , for every  $i \in \{1, \dots, t\}$  and  $j \in \{1, \dots, r\}$ . Hence,  $S[W] = (\bigcup_{i=1}^r B_i) \cup (\bigcup_{i=1}^t \{v_i\})$ . Therefore  $x \notin S[W]$ , which is a contradiction.  $\square$

**Corollary 23.** *If  $G$  is a graph of diameter two, then  $g(G) \leq s(G)$ .*

Now, from Theorem 6, Proposition 18 and Corollary 23 we obtain the following interesting result in which we give an infinite number of graphs  $G$  satisfying that  $g(G) \leq s(G)$ .

**Theorem 24.** *Let  $G$  be a connected graph of order  $n_1 \geq 2$  and let  $H$  be any non complete graph of order  $n_2$ . Then,*

$$g(G \odot H) \leq s(G \odot H).$$

*Proof.* By Theorem 6 we have that  $g(G \odot H) = ng(K_1 \odot H)$ . Since,  $K_1 \odot H$  has diameter two, by using Corollary 23 we have that  $g(K_1 \odot H) \leq s(K_1 \odot H)$ . Finally, by Proposition 18 we know that  $s(G \odot H) = n_1 n_2$ . Hence,

$$g(G \odot H) = n_1 g(K_1 \odot H) \leq n_1 s(K_1 \odot H) \leq n_1 n_2 = s(G \odot H).$$

$\square$

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